

FOUNDATIONS OF COMPUTATIONAL IMAGING  
FALL 2025  
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**Problem Set 1**

Issued: Tuesday, August 19, 2025

Due: Tuesday, August 26, 2025, 5:00 pm.

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**Problem 1** (Denoising). In the first chapter, we saw that signal models are central for solving inverse problems. Here, we consider a denoising problem and show that for a  $n$ -dimensional signal that lies in a  $k$ -dimensional subspace, we can remove an  $\frac{n-k}{n}$  fraction of additive Gaussian noise.

Consider a signal  $\mathbf{x}^* \in \mathbb{R}^n$  that lies in a  $k$ -dimensional subspace, and suppose we are given a noisy measurement

$$\mathbf{y} = \mathbf{x}^* + \mathbf{z},$$

where  $\mathbf{z}$  is zero-mean Gaussian noise with co-variance matrix  $(\sigma^2/n)\mathbf{I}$ . Let  $\mathbf{U} \in \mathbb{R}^{n \times k}$  be an orthonormal basis of the signal subspace. We denoise the signal by projecting the measurement onto the subspace, i.e., we consider the estimate  $\hat{\mathbf{x}} = \mathbf{U}\mathbf{U}^\top \mathbf{y}$ .

1. Show that

$$\mathbb{E} \left[ \|\hat{\mathbf{x}} - \mathbf{x}^*\|_2^2 \right] = \sigma^2 \frac{k}{n},$$

where expectation is over the random noise  $\mathbf{z}$ .

**Hint:** Recall that if  $\mathbf{V} \in \mathbb{R}^{n \times n}$  is a unitary matrix (i.e., a matrix with orthonormal columns) and  $\mathbf{z}$  has iid, zero-mean Gaussian entries, then  $\mathbf{V}\mathbf{z}$  has the same distribution as  $\mathbf{z}$ .

2. Does the estimator  $\hat{\mathbf{x}} = \mathbf{U}\mathbf{U}^\top \mathbf{y}$  remove more noise as the subspace dimension shrinks? And intuitively, do you think a better denoising algorithm exists?
3. Next, we study this denoising algorithm numerically (ideally with python in a jupyter notebook or Google colab using the library numpy; if you are not familiar with those, this exercise is a good exercise to familiarize yourself).

Generate a random  $k$ -dimensional subspace in  $\mathbb{R}^{1000}$ , and generate 500 random points in that subspace. Next, denoise each of those data points with the estimator above, and plot the average of the mean-squared error  $\|\hat{\mathbf{x}} - \mathbf{x}^*\|_2^2 / \|\mathbf{x}^*\|_2^2$  along with corresponding standard deviations as error bar for different values of  $k = 1, 100, 200, \dots, 1000$ .

We intentionally did not specify the method for generating a random subspace or for sampling points within it; you are encouraged to make a reasonable choice.